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A FURTHER GENERALIZATION OF KRALL'S JACOBI TYPE POLYNOMIALS

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A further generalization of Krall's Jacobi type polynomials*)

by

T.H. Koornwinder

ABSTRACT

We study orthogonal polynomials for which the weight function is a linear combination of the Jacobi weight function and two delta functions at 1 and -1. These polynomials can be expressed as ${}_4F_3$ hypergeometric functions and they satisfy second order differential equations. They include Krall's Jacobi type polynomials as special cases. The fourth order differential equation for the latter polynomials is derived in a more simple way.

KEY WORDS & PHRASES: *Jacobi type polynomials; orthogonal polynomials satisfying a second order differential equation; orthogonal polynomial eigenfunctions of a fourth order differential operator*

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*) This report will be submitted for publication elsewhere.

0. INTRODUCTION

The nonclassical orthogonal polynomials which are eigenfunctions of a fourth order differential operator were classified by H.L. KRALL [4], [5]. These polynomials were described in more details by A.M. KRALL [3]. The corresponding weight functions are special cases of the classical weight functions together with a delta function at the end point(s) of the interval of orthogonality. A number of A.M. Krall's results can be obtained in a more satisfactory way:

(a) Jacobi, Legendre and Laguerre type polynomials are connected with each other by quadratic transformations and a limit formula.

(b) The power series for the Jacobi type polynomials is of ${}_3F_2$-type.

(c) There is a pair of second order differential operators not depending on n which connect the Jacobi polynomials $P_n^{(\alpha,0)}(2x-1)$ and the Jacobi type polynomials $S_n(x)$. Combination of these two differentiation formulas yields the fourth order equation for $S_n(x)$.

It is the first purpose of the present paper to make these comments to [3]. The second purpose is to describe a more general class of Jacobi type polynomials, with weight function $(1-x)^\alpha(1+x)^\beta$ + linear combination of $\delta(x+1)$ and $\delta(x-1)$. They can be expressed in terms of Jacobi polynomials as $((a_nx+b_n)d/dx+c_n)P_n^{(\alpha,\beta)}(x)$ for certain coefficients a_n, b_n, c_n and their power series in $\frac{1}{2}(1-x)$ is of ${}_4F_3$ type. Finally, they satisfy a second order differential equation with polynomial coefficients depending on n, thus generalizing the known result for the Jacobi type polynomials $S_n(x)$ (cf. LITTLEJOHN & SHORE [6]) and providing further examples for HAHN's [2] general theory.

The general class of polynomials introduced here is remarkable because they are ${}_4F_3$'s, but I don't think they will have much further use. Anyhow, the formulas for these polynomials seem to lack the beauty of the classical Jacobi polynomials.

1. JACOBI POLYNOMIALS

We summarize the properties of Jacobi polynomials we need, cf. [1, §10.8].

Let $\alpha,\beta > -1$. *Jacobi polynomials* $P_n^{(\alpha,\beta)}(x)$ are orthogonal polynomials on the interval $[-1,1]$ with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ and with the normalization

$$P_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n! \tag{1.1}$$

Symmetry properties:

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x). \tag{1.2}$$

Differentiation formula:

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{1.3}$$

Rodrigues formula:

$$\begin{aligned} &(-1)^n 2^n n!\,(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) \\ &= (d/dx)^n((1-x)^{n+\alpha}(1+x)^{n+\beta}). \end{aligned} \tag{1.4}$$

Power series expansion:

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\;{}_2F_1\left(-n,n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\right) \\ &= \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k k!}\left(\frac{1-x}{2}\right)^k. \end{aligned} \tag{1.5}$$

Laguerre polynomials:

$$L_n^{\alpha}(x) := \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2\beta^{-1}x), \tag{1.6}$$

orthogonal on $[0,\infty)$ with respect to the weight function $e^{-x}x^{\alpha}$.

Differential equation:

$$\begin{aligned} &[(1-x^2)d^2/dx^2 + (\beta-\alpha-(\alpha+\beta+2)x)d/dx]P_n^{(\alpha,\beta)}(x) \\ &= -n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x). \end{aligned} \tag{1.7}$$

2. DEFINITION

Fix $M,N \geq 0$ and $\alpha,\beta > -1$. For $n = 0,1,2,\ldots$ define

$$(2.1)\qquad P_n^{\alpha,\beta,M,N}(x) := ((\alpha+\beta+1)_n/n!)^2 \cdot$$
$$\cdot[(\alpha+\beta+1)^{-1}(B_nM(1-x) - A_nN(1+x))d/dx + A_nB_n]P_n^{(\alpha,\beta)}(x),$$

where

$$(2.2)\qquad A_n := \frac{(\alpha+1)_n n!}{(\beta+1)_n(\alpha+\beta+1)_n} + \frac{n(n+\alpha+\beta+1)M}{(\beta+1)(\alpha+\beta+1)},$$

$$(2.3)\qquad B_n := \frac{(\beta+1)_n n!}{(\alpha+1)_n(\alpha+\beta+1)_n} + \frac{n(n+\alpha+\beta+1)N}{(\alpha+1)(\alpha+\beta+1)}.$$

The case $\alpha+\beta+1 = 0$ must be understood by continuity in α, β. By using (1.1) and (1.3) we find

$$(2.4)\qquad P_n^{\alpha,\beta,M,N}(1) = \frac{(\alpha+1)_n}{n!} + \frac{(\beta+1)_n(\alpha+\beta+2)_n nM}{n!\,n!\,(\beta+1)}.$$

From (1.2) we have the symmetry

$$(2.5)\qquad P_n^{\alpha,\beta,M,N}(-x) = (-1)^n P_n^{\beta,\alpha,N,M}(x).$$

3. ORTHOGONALITY

Define the measure μ on $[-1,1]$ by

$$(3.1)\qquad \int_{-1}^{1} f(x)d\mu(x) := \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta}dx +$$
$$+ Mf(-1) + Nf(1), \qquad f \in C([-1,1]).$$

THEOREM 3.1. *The polynomials* $P_n^{\alpha,\beta,M,N}(x)$ *are orthogonal polynomials on the interval* $[-1,1]$ *with respect to the measure* μ *and with the normalization* (2.4).

PROOF. By (2.1) and (2.3), $P_n^{\alpha,\beta,M,N}(x)$ is a polynomial of degree $\le n$, not identically zero.

In order to prove the orthogonality first assume $n \ge 2$. Observe that the polynomials $(1+x)^k(1-x)^{n-k-1}$ $(k = 0,1,\ldots,n-1)$ form a basis for the space of polynomials of degree $\le n-1$. If $1 \le k \le n-2$ then

$$\int_{-1}^{1} P_n^{\alpha,\beta,M,N}(x)(1-x)^{n-k-1}(1+x)^k d\mu(x) = 0$$

by integration by parts and the orthogonality property of Jacobi polynomials. Now consider $k = 0$:

$$I := \int_{-1}^{1} P_n^{\alpha,\beta,M,N}(x)(1-x)^{n-1} d\mu(x).$$

The continuous part of μ yields a contribution

$$I_1 := \frac{\Gamma(\alpha+\beta+1)(n+\alpha+\beta+1)B_n M}{2^{\alpha+\beta+3-n}\Gamma(\alpha+1)\Gamma(\beta+1)} \cdot \int_{-1}^{1} P_{n-1}^{(\alpha+1,\beta+1)}(x)(1-x)^{\alpha+1}(1+x)^{\beta}dx,$$

where we used (1.3) and the orthogonality property of Jacobi polynomials. Now substitute (1.4), integrate by parts and evaluate the resulting beta integral:

$$I_1 = (-1)^{n-1}2^{n-1}(\alpha+1)_n B_n M/(\alpha+\beta+1)_n.$$

The discrete part of μ yields a contribution $-I_1$ to I (use (1.5), (1.2) and (1.1)) so $I = 0$. The case $k = n-1$ follows from the case $k = 0$ by (2.5).

Finally consider the case $n = 1$. By (1.5) we have

$$P_1^{(\alpha,\beta)}(x) = (\alpha+1) - \tfrac{1}{2}(\alpha+\beta+2)(1-x),$$

so

$$P_1^{\alpha,\beta,M,N}(x) = -\tfrac{1}{2}(\alpha+1)(\alpha+\beta+1)B_1(1-x) + \tfrac{1}{2}(\beta+1)(\alpha+\beta+1)A_1(1+x).$$

Hence

$$\int_{-1}^{1} P_1^{\alpha,\beta,M,N}(x)\,d\mu(x) = 0$$

by evaluating the beta integrals. □

4. SPECIAL CASES

Of course:

$$(4.1)\qquad P_n^{\alpha,\beta,0,0}(x) = P_n^{(\alpha,\beta)}(x).$$

Next we have

$$(4.2)\qquad \begin{aligned} &P_n^{\alpha,\beta,M,0}(x)\\ &= [1 + \frac{M(\beta+1)_n(\alpha+\beta+1)_n}{(\alpha+1)_n n!\,(\alpha+\beta+1)}\,((1-x)\frac{d}{dx} + \frac{n(n+\alpha+\beta+1)}{\beta+1})]P_n^{(\alpha,\beta)}(x), \end{aligned}$$

$$(4.3)\qquad \begin{aligned} S_n(x) &= MP_n^{\alpha,0,(\alpha+1)/M,0}(2x-1)\\ &= ((1-x)d/dx + n(n+\alpha+1) + M)P_n^{(\alpha,0)}(2x-1), \end{aligned}$$

where $S_n(x)$ are KRALL's [3, § 16,17] Jacobi type polynomials, orthogonal with respect to the measure $((1-x)^{\alpha} + M^{-1}\delta(x))dx$ on $[0,1]$.

Furthermore,

$$(4.4)\qquad \begin{aligned} P_n^{\alpha,\alpha,M,M}(x) &= \left(1 + \frac{M(2\alpha+2)_n n}{(\alpha+1)n!}\right)\cdot\\ &\cdot[1 + \frac{M(2\alpha+1)_n}{n!\,(2\alpha+1)}\,(-2x\frac{d}{dx} + \frac{n(n+2\alpha+1)}{\alpha+1})]P_n^{(\alpha,\alpha)}(x), \end{aligned}$$

$$(4.5)\qquad \begin{aligned} P_n^{(\alpha)}(x) &= \frac{\alpha^2}{\alpha+\frac{1}{2}n(n+1)}\,P_n^{0,0,1/(2\alpha),1/(2\alpha)}(x)\\ &= (-x\,d/dx + \alpha + \tfrac{1}{2}\,n(n+1))P_n(x), \end{aligned}$$

where $P_n^{(\alpha)}(x)$ are KRALL's [3, § 4.5] Legendre type polynomials, orthogonal with respect to the measure $\frac{1}{2}(\alpha+\delta(x-1) + \delta(x+1))dx$ on $[-1,1]$.

By using Theorem 3.1 we obtain the quadratic transformations

$$(4.6) \qquad \frac{P_{2n}^{\alpha,\alpha,M,M}(x)}{P_{2n}^{\alpha,\alpha,M,M}(1)} = \frac{P_n^{\alpha,-\frac{1}{2},0,2M}(2x^2-1)}{P_n^{\alpha,-\frac{1}{2},0,2M}(1)},$$

$$(4.7) \qquad \frac{P_{2n+1}^{\alpha,\alpha,M,M}(x)}{P_{2n+1}^{\alpha,\alpha,M,M}(1)} = \frac{xP_n^{\alpha,\frac{1}{2},0,(4\alpha+6)M}(2x^2-1)}{P_n^{\alpha,\frac{1}{2},0,(4\alpha+6)M}(1)}.$$

In particular, these formulas connect Krall's Legendre and Jacobi type polynomials with each other.

$$(4.8) \qquad \begin{aligned} L_n^{\alpha,N}(x) &:= \lim_{\beta\to\infty} P_n^{\alpha,\beta,0,N}(1-2\beta^{-1}x) \\ &= [1 + \frac{N(\alpha+1)_n}{n!}(\frac{d}{dx} + \frac{n}{\alpha+1})]L_n(x), \end{aligned}$$

orthogonal polynomials on the interval $[0,\infty)$ with respect to the measure $((\Gamma(\alpha+1))^{-1}e^{-x}x^{\alpha} + N\delta(x))dx$ on the interval $[0,\infty)$ and with the normalization $L_n^{\alpha,N}(0) = (\alpha+1)_n/n!$ (cf. (1.6), (2.5), (4.2) and Theorem 3.1).

$$(4.9) \qquad R_n(x) = RL_n^{0,R^{-1}}(x),$$

where $R_n(x)$ are KRALL's [3, § 10,11] Laguerre type polynomials, orthogonal with respect to the measure $(e^{-x} + R^{-1}\delta(x))dx$ on $[0,\infty)$.

5. EXPRESSION AS HYPERGEOMETRIC SERIES

By (1.5) and (2.1) we have

$$\begin{aligned} &\frac{n!n!n!}{(\alpha+1)_n(\alpha+\beta+1)_n(\alpha+\beta+1)_n} P_n^{\alpha,\beta,M,N}(1-2x) \\ &= [(\alpha+\beta+1)^{-1}(-B_nMx + A_nN(1-x))d/dx + A_nB_n]\cdot \\ &\quad (\sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k\, k!} x^k). \end{aligned}$$

By straightforward calculations we obtain

$$(5.1)\qquad \frac{P_n^{\alpha,\beta,M,N}(1-2x)}{P_n^{\alpha,\beta,M,N}(1)} = \frac{(\alpha+1)_n(\alpha+\beta+1)_n}{(\alpha+1)(\beta+1)_n n!A_n}\ \cdot$$

$$\cdot \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+2)_k\ k!}\ [-MB_n(\alpha+\beta+1)^{-1}k^2 +$$

$$+ (NA_n(\alpha+\beta+1)^{-1}\beta - MB_n(\alpha+\beta+1)^{-1}(\alpha+1) + A_nB_n)k +$$

$$+ \frac{(\alpha+1)(\beta+1)_n n!}{(\alpha+1)_n(\alpha+\beta+1)_n} A_n]x^k.$$

For $M,N > 0$ this becomes

$$(5.2)\qquad \frac{P_n^{\alpha,\beta,M,N}(1-2x)}{P_n^{\alpha,\beta,M,N}(1)} = {}_4F_3\left(\left.\begin{matrix}-n,n+\alpha+\beta+1,-a_n+1,b_n+1\\ \alpha+2,-a_n,b_n\end{matrix}\right| x\right),$$

where $a_n > n$, $b_n > 0$ and

$$a_nb_n = \frac{(\alpha+1)(\alpha+\beta+1)(\beta+1)_n n!A_n}{(\alpha+1)_n(\alpha+\beta+1)_n MB_n},$$

$$a_n - b_n = \beta NM^{-1}A_nB_n^{-1} + (\alpha+\beta+1)M^{-1}A_n - \alpha - 1.$$

For $M = 0$, $N \neq 0$:

$$(5.3)\qquad \frac{P_n^{\alpha,\beta,0,N}(1-2x)}{P_n^{\alpha,\beta,0,N}(1)} = {}_3F_2\left(\left.\begin{matrix}-n,n+\alpha+\beta+1,c_n+1\\ \alpha+2,c_n\end{matrix}\right| x\right),$$

where

$$c_n = \frac{(\alpha+1)(\beta+1)_n\ n!}{(N(\alpha+\beta+1)^{-1}\beta+B_n)(\alpha+1)_n(\alpha+\beta+1)_n}.$$

For $N = 0$, $M \neq 0$:

$$\frac{P_n^{\alpha,\beta,M,0}(1-2x)}{P_n^{\alpha,\beta,M,0}(1)} = {}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+1, -(\alpha+\beta+1)M^{-1}A_n+1\\ \alpha+1, -(\alpha+\beta+1)M^{-1}A_n\end{matrix}\middle| x\right). \tag{5.4}$$

Combination of (4.3), (2.5) and (5.3) yields KRALL's power series expansion [3, § 16]. Combination of (4.6), (4.7), (2.5) and (5.4) yields power series expansion in x for $P_n^{\alpha,\alpha,M,M}(x)$, cf. [3, § 4].

6. SECOND ORDER DIFFERENTIAL EQUATION

Observe that

$$\begin{aligned}(B_nM - A_nN - (B_nM + A_nN)x)^2\ [(1-x^2)d^2/dx^2 + (\beta-\alpha-(\alpha+\beta+2)x)d/dx] +\\ + (\alpha+\beta+1)A_nB_nb_n(x)\\ = (a_n(x)d/dx + b_n(x))\ [B_nM-A_nN - (B_nM+A_nN)x)d/dx+(\alpha+\beta+1)A_nB_n],\end{aligned}$$

where

$$a_n(x) := (B_nM-A_nN-(B_nM+A_nN)x)(1-x^2), \tag{6.1}$$

$$\begin{aligned}b_n(x) := (\alpha+\beta+1)(B_nM+A_nN+A_nB_n)x^2+2((\alpha+1)A_nN-(\beta+1)B_nM)x +\\ + (\beta-\alpha+1)B_nM + (\alpha-\beta+1)A_nN - A_nB_n(\alpha+\beta+1).\end{aligned} \tag{6.2}$$

Also put

$$\begin{aligned}c_n(x) := A_nB_nb_n(x) +\\ -n(n+\alpha+\beta+1)(\alpha+\beta+1)^{-1}(B_nM-A_nN-(B_nM+A_nN)x)^2.\end{aligned} \tag{6.3}$$

Then, by use of (1.7) and (2.1) it follows that

$$(6.4)\qquad (a_n(x)\,d/dx + b_n(x))P_n^{\alpha,\beta,M,N}(x) = ((\alpha+\beta+1)_n/n!)^2 c_n(x)P_n^{(\alpha,\beta)}(x).$$

Combination of (6.4) and (2.1) yields

THEOREM 6.1. $P_n^{\alpha,\beta,M,N}(x)$ *satisfies a second order differential equation with polynomial coefficients depending on* n *but of bounded degree.*

See HAHN [2] for a more general study of orthogonal polynomials with this property. LITTLEJOHN & SHORE [6] derive second order differential equations for the polynomials (4.3), (4.5), (4.9) in a different, more complicated way.

7. FOURTH ORDER DIFFERENTIAL EQUATION FOR KRALL'S JACOBI TYPE POLYNOMIALS

Fix $\alpha > -1$ and $M > 0$. Let $S_n(x)$ be defined by (4.3). Combination of (4.3) and (1.7) yields

$$(7.1)\qquad S_n(x) = [x(x-1)d^2/dx^2 + (\alpha+1)x\,d/dx + M]P_n^{(\alpha,0)}(2x-1).$$

Observe that, for arbitrary polynomials f, g, we have

$$(7.2)\qquad \begin{aligned} &\int_0^1 g(x)[x(x-1)d^2/dx^2 + (\alpha+1)xd/dx + M]f(x)\ \ ((1-x)^\alpha + M^{-1}\delta(x))dx \\ &= \int_0^1 f(x)[x(x-1)/d^2/dx^2 + ((\alpha+3)x-2)d/dx+M+\alpha+1]g(x)\ \ (1-x)^\alpha dx. \end{aligned}$$

Formulas (7.1), (7.2) and the orthogonality properties of $S_n(x)$ and $P_n^{(\alpha,0)}(2x-1)$ imply:

$$(7.3)\qquad \begin{aligned} &((n+\alpha+1)(n+1)+M)(n(n+\alpha)+M)P_n^{(\alpha,0)}(2x-1) \\ &= [x(x-1)\frac{d^2}{dx^2} + ((\alpha+3)x-2)\frac{d}{dx} + M+\alpha+1]S_n(x), \end{aligned}$$

where the coefficient of $P_n^{(\alpha,0)}(2x-1)$ is obtained by comparing the coefficients of x^n at both sides of (7.3). Combination of (7.1) and (7.3) yields

THEOREM 7.1. *The polynomials* $S_n(x)$ *are eigenfunctions of a fourth order differential operator with polynomial coefficients not depending on* n.

A calculation leads to the explicit form of KRALL's [3, § 14] differential equation.

8. LITTLEJOHN'S ORTHOGONAL POLYNOMIALS

After completion of the manuscript I received a paper by LITTLEJOHN [7], where he proves that the polynomials $P_n^{0,0,M,N}(x)$ (notation of the present paper) are eigenfunctions of a sixth order differential operator. The techniques of Section 7 also apply to this case and would lead to an eighth order differential operator.

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